On the Emergence of Delta-Vega Hedging in the Black and Scholes Model

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ETH Risk Day 2015

Zürich, September 11, 2015

Outline

Introduction

Black and Scholes

- Riskless bank account: $B \equiv 1$.
- Risky stock S with risk-neutral dynamics

$$\frac{\mathrm{d}S_t}{S_t} = \sigma \,\mathrm{d}W_t$$

  with a **constant** spot volatility $\sigma > 0$.
- Replication argument gives option prices and hedging strategies.

Delta hedging

- Agent has sold an option with payoff V(S_T) for the BS value $\mathcal{V}(0, S_0, \sigma)$.
- Hedges her exposure by trading in stock and bank account.
- **Delta hedge:** hold $\theta_t = \frac{\partial \mathcal{V}}{\partial S}(t, S_t, \sigma)$ shares at time t.
- Delta-hedged portfolio is **delta-neutral**:

$$\left.\begin{array}{ccccccc} \#\text{shares} & \times & \text{delta(stock)} & + & \#\text{options} & \times & \text{delta(option)} \\ \theta_t & \times & 1 & + & (-1) & \times & \frac{\partial \mathcal{V}}{\partial S}(t, S_t, \sigma) \end{array}\right\} = 0$$

- Which volatility σ to use? Statistical vs. "implied" approach.

Implied volatility

What if there is a **liquidly traded** call option?

- Call price at time t is quoted in terms of its **implied volatility** Σ_t solving

$$\mathcal{C}(t, S_t, \Sigma_t) = \text{market price of call at time } t,$$

  where $\mathcal{C}(t, S, \Sigma)$ is the BS call value for volatility Σ.

- If the BS model is "correct", then $\Sigma_t = \sigma$ for all t.

Delta-vega hedging

- Use call for hedging.
- **Vega**: sensitivity of option price wrt. implied volatility

$$\textbf{call vega} \quad \mathcal{C}_\Sigma = \frac{\partial \mathcal{C}}{\partial \Sigma} \qquad \textbf{option vega} \quad \mathcal{V}_\Sigma = \frac{\partial \mathcal{V}}{\partial \Sigma}$$

$$\textbf{call delta} \quad \mathcal{C}_S = \frac{\partial \mathcal{C}}{\partial S} \qquad \textbf{option delta} \quad \mathcal{V}_S = \frac{\partial \mathcal{V}}{\partial S}$$

- **Delta-vega hedging**: choose ϕ and θ such that total portfolio is

$$\underset{\phi\mathcal{C}_\Sigma - \mathcal{V}_\Sigma = 0}{\textbf{vega-neutral}} \quad \text{and} \quad \underset{\theta + \phi\mathcal{C}_S - \mathcal{V}_S = 0}{\textbf{delta-neutral}} \; .$$

- Justification?
- Again, which volatility to use?

Efficient markets?

What if the implied volatility Σ_t moves away from σ?

“Hedge fund perspective”

Market is wrong!

$\longrightarrow$ Keep your model
(and exploit “mispricing”)

“Risk management perspective”

Market is always right!

$\longrightarrow$ **Recalibrate model**
and hedge with volatility Σ_t.

Results in a nutshell

Problem:

- In practise:
  - model parameters are frequently recalibrated to market prices of liquidly traded options,
  - sensitivities with respect to model parameters are hedged.
- This is inconsistent with the assumption of constant parameters.
- Model uncertainty cannot be quantified consistently.

We find:

- Recalibrated delta-vega hedging is almost optimal for agents with small uncertainty aversion who believe in market efficiency.
- Impact of uncertainty aversion can be quantified in terms of the disparity between the vegas, gammas, vannas, and volgas of the liquidly traded and the (exotic) option.

Problem formulation

Agent's beliefs

Consider an agent with the following beliefs:

1. **Model uncertainty:** True dynamics of stock and the liquidly traded option are **not known precisely**.

2. **Market efficiency:** The market prices liquidly traded options correctly.

3. **Moderate risk and uncertainty aversion:**
   - My **reference model** is my **best guess** for the true dynamics (but could be incorrect).
   - I take alternative models less seriously the more they deviate from my reference model.
   - I am risk-averse.

How to describe the preferences of such an agent?

Variational preferences

- Maccheroni/Marinacci/Rustichini '06: Decision-theoretic axioms suggest representation

$$\inf_P \left(E^P \left[U(Y) \right] + \alpha(P) \right)$$

- Utility function U models **risk aversion** in a given model.
- Penalty functional α describes **uncertainty aversion**.
- Classical approach: infinite penalty for all but one model P.
- Worst-case approach: no penalty for a class $\mathfrak{P}$ of plausible models; infinite penalty otherwise.
  $\to$ Avellaneda/Levy/Paras '95, Lyons '95, ...
- "Smooth" alternatives?

Models

- **Plausible models**: Probability measures P under which (S, Σ) has dynamics of the form

$$\mathrm{d}S_t = S_t \sigma_t^P \,\mathrm{d}W_t^0,$$

$$\mathrm{d}\Sigma_t = \nu_t^P \,\mathrm{d}t + \eta_t^P \,\mathrm{d}W_t^0 + \sqrt{\xi_t^P} \,\mathrm{d}W_t^1,$$

  and "call" price $C_t := \mathcal{C}(t, S_t, \Sigma_t)$ is drift-less (no-arbitrage principle).
  $\rightarrow$ Lyons '97, Schönbucher '99, ...

- **Reference model**: Black–Scholes
  - corresponds to $\nu^P = \eta^P = \xi^P = 0$ and $\sigma^P = \Sigma$.

    "The future implied volatility stays at its currently observed level."

  - benchmark case; general approach extends to more realistic reference models.

Hedging problem

- P&L with dynamic trading in stock and “call”:

$$Y_T^{\theta,\phi} = V_0 + \int_0^T \theta_u \,\mathrm{d}S_u + \int_0^T \phi_u \,\mathrm{d}C_u - \mathsf{V}(S_T)$$

- Hedging problem:

$$v(\psi) = \sup_{\theta,\phi} \inf_P \left(E^P \left[U(Y_T^{\theta,\phi}) \right] + \alpha^\psi(P) \right)$$

- Penalty functional $\alpha^\psi(P) \geq 0$: describes plausibility of a model P, zero for BS model and positive for alternative models.

Penalty functional

- Penalise “mean-square” deviations from the reference BS model:

$$\alpha^{\psi}(P) = \frac{1}{\psi} E^{P} \left[\int_0^T U'(Y_t^{\theta,\phi}) \left\{ (\nu_t^P)^2 + (\sigma_t^P - \Sigma_t)^2 + (\eta_t^P)^2 + (\xi_t^P)^2 \right\} \mathrm{d}t \right]$$

- Recall:

  ν^P: drift of implied volatility
  σ^P: spot volatility
  η^P: correlated volatility of implied volatility
  ξ^P: uncorrelated squared volatility of implied volatility

- $\psi > 0$ measures magnitude of uncertainty aversion.

- To obtain explicit results: study limit $\psi \downarrow 0$.

Results

Almost optimality of delta-vega hedging

- Value has asymptotic expansion of the form

$$v(\psi) = U(0) - U'(0)\widetilde{w}\psi + o(\psi) \quad \text{as} \quad \psi \downarrow 0.$$

- Dynamically recalibrated delta-vega hedge is optimal at the leading-order $O(\psi)$:
  - #“calls” $\phi^\star = \mathcal{V}_\Sigma / \mathcal{C}_\Sigma$ neutralises vega,
  - #shares $\theta^\star = \mathcal{V}_S - \phi^\star \mathcal{C}_S$ neutralises delta.

- Correction term $\widetilde{w}$ is determined by

$$\begin{aligned} \textbf{net gamma} &= \phi^\star \mathcal{C}_{SS} - \mathcal{V}_{SS} \\ \textbf{net vanna} &= \phi^\star \mathcal{C}_{S\Sigma} - \mathcal{V}_{S\Sigma} \\ \textbf{net volga} &= \phi^\star \mathcal{C}_{\Sigma\Sigma} - \mathcal{V}_{\Sigma\Sigma} \end{aligned}$$

Value expansion

- Recall: $v(\psi) = U(0) - U'(0)\widetilde{w}\psi + o(\psi)$.
- $\widetilde{w}$ has probabilistic representation

$$\widetilde{w} = \frac{1}{2} E\left[\int_0^T \widetilde{g}(t, S_t, \Sigma_0)\,\mathrm{d}t\right]$$

$$\begin{aligned} \widetilde{g}(t,S,\Sigma) = \Sigma S^2 \left(\mathcal{V}_{SS} - \phi^\star \mathcal{C}_{SS}\right)\widetilde{\sigma} \quad & \text{–(net gamma)} \times \textbf{spot vol deviation} \\ +\Sigma S \left(\mathcal{V}_{S\Sigma} - \phi^\star \mathcal{C}_{S\Sigma}\right)\widetilde{\eta} \quad & \text{–(net vanna)} \times \textbf{correlated vol} \\ +\tfrac{1}{2}\left(\mathcal{V}_{\Sigma\Sigma} - \phi^\star \mathcal{C}_{\Sigma\Sigma}\right)\widetilde{\xi} \quad & \text{–(net volga)} \times \textbf{uncorrelated vol}^2. \end{aligned}$$

- $\widetilde{\sigma}$, $\widetilde{\eta}$, $\widetilde{\xi}$ are related to deviations of "subjective worst-case model" P^ψ from reference BS model:

spot vol	correlated vol of IV	uncorrelated vol^2 of IV
$\sigma^{P^\psi} \approx \Sigma + \widetilde{\sigma}\psi$	$\eta^{P^\psi} \approx \widetilde{\eta}\psi$	$\xi^{P^\psi} \approx \widetilde{\xi}\psi$

Indifference prices

- Bid and ask indifference prices for option V:

$$p_b(\psi) = V_0 - \widetilde{w}\psi + o(\psi) \quad \text{and} \quad p_a(\psi) = V_0 + \widetilde{w}\psi + o(\psi)$$

- $\widetilde{w}\psi$ is “compensation” for model uncertainty, independent of the utility function.
- Sanity check: $\widetilde{w} = 0$ if C is a call and V is a put with matching strike and maturity (model-free hedge by put-call parity).

Illustrations

Log contract

- Payoff: $\log S_{T'}$.
- Can be replicated statically by a continuum of calls and puts with the same maturity.
- Simple BS value and Greeks: $\mathcal{C}(t, S, \Sigma) = \log S - \frac{1}{2}\Sigma^2(T' - t)$,

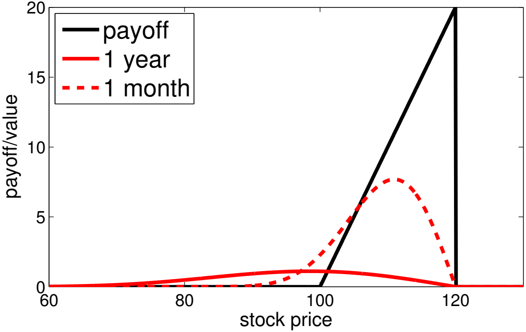
vega	**gamma**	**vanna**	**volga**
$\mathcal{C}_\Sigma = -\Sigma(T' - t)$,	$\mathcal{C}_{SS} = -1/S^2$,	$\mathcal{C}_{S\Sigma} = 0$,	$\mathcal{C}_{\Sigma\Sigma} = -(T' - t)$.

- Good proxy for a range of traded calls/puts with different strikes, but same maturity.

We assume now that the liquidly traded option is a log contract with maturity $T' \geq T$.

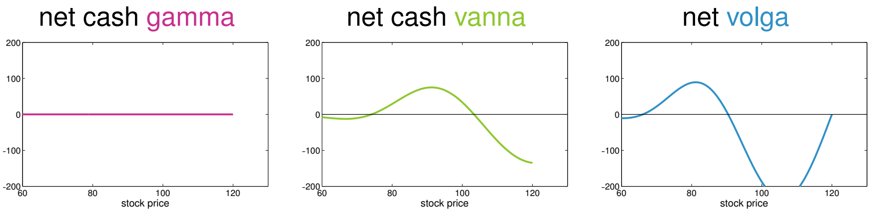
Up-and-out call

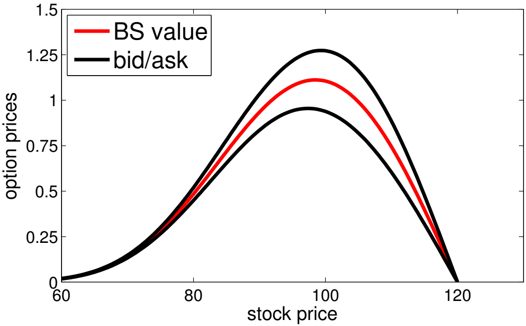
- Call option (strike K) which becomes worthless if the stock hits a barrier $B > K$ at any time before maturity T.



- Very high (absolute) Greeks in the vicinity of the barrier and close to maturity.

Hedging an up-and-out call



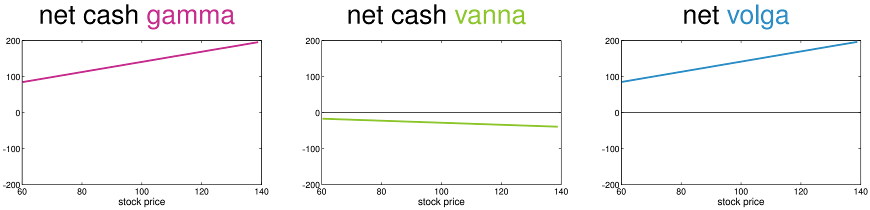


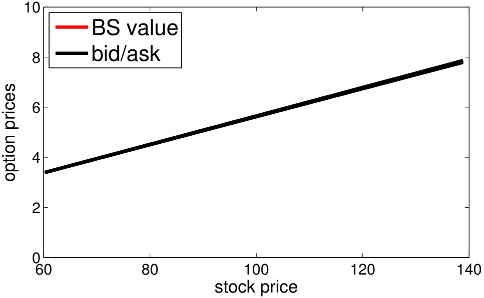
- Hedge an up-and-out call with with strike 100 , barrier 120, and maturity 1y.
- Maturity of log contract: $T' =$1y
- All plots at $t = 0$.

Forward-start call

- Call option whose strike is set at some future date $T_{\mathrm{reset}} \leq T$.
- Payoff: $(S_T - S_{T_{\mathrm{reset}}})^+$.
- Gives exposure to spot volatility on $[T_{\mathrm{reset}}, T]$.
- Before T_{reset}: **gamma** $= 0$ but **vega** > 0.
  $\rightarrow$ delta-vega hedged portfolio will **not** be gamma-neutral on $[0, T_{\mathrm{reset}}]$.

Hedging a forward-start call

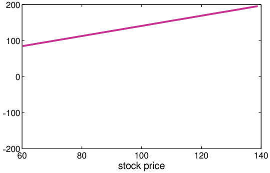




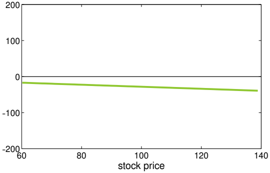
- Hedge forward-start call with reset in 6m and maturity 1y.
- Maturity of log contract: $T' =$1y
- All plots at $t = 0$.

Hedging a forward-start call

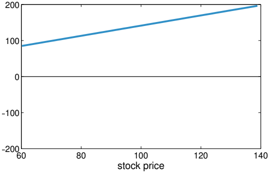
net cash gamma

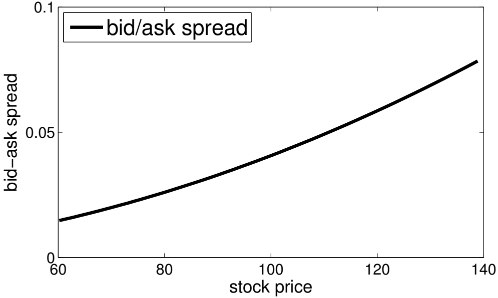


net cash vanna



net volga





- Hedge forward-start call with reset in 6m and maturity 1y.
- Maturity of log contract: $T' =$1y
- All plots at $t = 0$.

Summary

- Hedging of an (exotic) option under **model uncertainty** with dynamic trading in stock and “call”.
- Explicit formulas in the limit for small uncertainty aversion.
- Impact of uncertainty depends on disparity between the vegas, gammas, vannas, and volgas of the “call” and the (exotic) option; independent of risk aversion.
- Dynamically recalibrated **delta-vega hedging** is leading-order optimal.

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